On the Theoretical Equivalence of

Differently Proposed Ensemble/3D-Var Hybrid Analysis Schemes

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Abstract

Hybrid ensemble/3-dimensional variational analysis schemes incorporate flow-dependent, ensemble-estimated background-error covariances into the 3-dimensional variational (3D-Var) framework. Typically the 3D-Var background-error covariance estimate is assumed to be stationary, nearly homogeneous and isotropic. A hybrid scheme can be achieved by directly replacing the background-error covariance term in the cost function by a linear combination of the original background-error covariance with the ensemble covariance or through augmenting the state vector with another set of control variables preconditioned upon the square root of the ensemble covariance. These differently proposed hybrid schemes are proved to be equivalent. The latter framework may be a simpler way to incorporate ensemble information into operational 3-dimensional variational schemes, where the preconditioning is performed with respect to the background term.

1. Introduction

Present 3-dimensional variational data assimilation schemes (3D-var; e.g., Parrish and Derber 1992; Courtier et al. 1998; Gauthier et al. 1998) typically assume that the background-error covariances are stationary, and nearly homogeneous and isotropic, while in fact the error covariances may vary substantially with the flow of the day. Several approaches have been proposed to relax these assumptions in 3D-Var. Fisher and Courtier (1995) suggested an approach to developing flow-dependent background covariances in the variational data assimilation, in which the leading eigenvectors of the background error covariance matrix are explicitly estimated. Techniques are also being developed to include some spatial inhomogeneity and anisotropy in 3D-Var (e.g., Desroziers 1997; Riishøjgaard 1998; Purser et al. 2003; Wu et al. 2002). Another different approach is to blend in flow-dependent error covariances estimated by an ensemble into the variational framework (Barker 1999; Hamill and Snyder 2000; Lorenc 2003; Etherton and Bishop 2004; Buehner 2005; Wang et al. 2005). These latter methods are known as hybrid ensemble-variational schemes, or more simply here as hybrid schemes. In this paper we focus on discussing the differently proposed hybrid ensemble-variational schemes.

A hybrid scheme was proposed and tested by Hamill and Snyder (2000), hereafter "HS00." In that study, the background-error covariance was explicitly replaced by a linear combination of the 3D-Var background-error covariance and the sample ensemble covariance. Each member was then updated variationally with perturbed observations. Parallel assimilations and forecasts were cycled forward, as in a traditional ensemble Kalman filter scheme (e.g., Houtekamer and Mitchell 1998, 2001; Houtekamer et al. 2005). Later, Etherton and Bishop (2004) and Wang et al. (2005) provided an implementation of HS00, where the ensemble perturbations were updated by the ensemble transform Kalman filter (ETKF; Bishop et al. 2001, Wang and Bishop 2003, Wang et al. 2004) and the background error covariance for updating the mean state was given by an explicit sum of the ETKF ensemble covariance and the standard 3DVAR covariance.

Lorenc (2003, hereafter L03) proposed another form of the hybrid variational scheme for updating the state, where the control variables in the cost function were augmented by another set of control variables, preconditioned upon the square root of the ensemble covariance. He also showed how a localizing Schur product, which will reduce the effects of sampling error in the ensemble covariances, could be implemented in the variational framework with preconditioning. Buehner (2005; hereafter B05) adopted a hybrid framework similar to L03 to incorporate the ensemble covariance output from the ensemble Kalman filter (EnKF) into the 3D-Var system. Another implementation of the Schur product for covariance localization was also proposed by B05.

Hybrid schemes present a possible alternative to the canonical ensemble data assimilation schemes (e.g., Evensen 1994; Burgers et al. 1998; Anderson 2001; Bishop et al. 2001; Whitaker and Hamill 2002; Whitaker and Hamill 2005; Snyder and Zhang 2003; Zhang et al. 2004; Ott et al. 2004; Szunyogh et al. 2005; Houtekamer and Mitchell 1998, 2001,2005; Houtekamer et al. 2005). Unlike the ensemble data assimilation schemes that adopt a framework completely different from the existing variational scheme, the hybrid schemes begin with existing variational systems and thus can be built as incremental changes to the existing variational codes. Hybrids may be less computationally expensive than the ensemble data assimilation schemes. Since many of the ensemble data assimilation schemes assimilate observations serially, their computational expense typically scales not only with the number of ensemble members and the dimension of the model state, but also with the number of observations. This may be a concern for operational applications, as the number of observations is huge and still growing with each passing year. In comparison, the computational expense of variational techniques currently used in operational centers do not scale linearly with the number of observations. Another potential advantage of hybrids is the ease of applying variational quality control (L03). Consequently, if hybrid methods can achieve much of the potential error reduction of these ensemble filters (Wang et al. 2005), then they may provide an attractive alternative for operational centers where variational data assimiation is established and ensemble forecasts are available or a suitable and efficient method can be found to form the background ensemble.

The hybrid schemes proposed by HS00 and by L03 and B05 differ in the way that they incorporate the ensemble-covariance information into the cost function, though L03 and B05 state without proof that the schemes are equivalent or similar. The purpose of this note is to provide a proof that the variational state update steps for the two hybrid schemes proposed by HS00, and by L03 and B05 are mathematically equivalent. Also, we show that the methods that L03 and B05 proposed to implement a localizing Schur product in the variational framework with preconditioning are equivalent. Because augmenting the control vector as in L03 and B05 may be easier to implement within those variational systems in which the preconditioning is with respect to the background term, this may provide a convenient pathway for the incorporation of ensemble information into many operational analysis schemes.

Below, section 2 will provide a detailed proof of the equivalence of the two proposed hybrid schemes. Section 3 discusses operational applications of the hybrid schemes. Section 4 concludes the paper.

2. Proof of equivalence of the hybrid schemes

In HS00, the cost function associated with the hybrid ensemble-3D-Var background-error covariance is

$$J = \frac{1}{2}\left(\mathbf{x}-\mathbf{x}^b\right)^T \mathbf{B}^{-1}\left(\mathbf{x}-\mathbf{x}^b\right)+\frac{1}{2}\left(H\left(\mathbf{x}\right)-\mathbf{y}\right)^T \mathbf{R}^{-1}\left(H\left(\mathbf{x}\right)-\mathbf{y}\right), \tag{1}$$

where $\mathbf{x}^b$ is a vector of background forecast, $\mathbf{y}$ contains the observations, $\mathbf{R}$ is the observation error covariance matrix, and H is the operator mapping from the model space to the observation space. The hybrid background-error covariance matrix $\mathbf{B}$ is given by the linear combination of the 3D-Var covariance matrix $\mathbf{B}_1$ and the ensemble covariance $\mathbf{B}_2$, i.e.,

$$\mathbf{B} = \alpha_1 \mathbf{B}_1 + \alpha_2 \mathbf{B}_2, \tag{2}$$

where α_1 and α_2 are the scalar linear combination coefficients. In HS00, $\alpha_1 = 1-\alpha_2$. Note covariance localization can be applied on the ensemble covariance $\mathbf{B}_2$ through a Schur, or element by element, product with a compactly supported correlation matrix (e.g., Houtekamer and Mitchell 2001). Further defining the analysis increment as $\Delta\mathbf{x} = \mathbf{x}-\mathbf{x}^b$, then (1) becomes

$$J = \frac{1}{2}(\Delta\mathbf{x})^T \mathbf{B}^{-1}(\Delta\mathbf{x}) + \frac{1}{2}\left(H\left(\mathbf{x}^b + \Delta\mathbf{x}\right) - \mathbf{y}\right)^T \mathbf{R}^{-1}\left(H\left(\mathbf{x}^b + \Delta\mathbf{x}\right) - \mathbf{y}\right). \quad (3)$$

This is the quadratic minimization problem solved in the "inner loop" of incremental variational schemes. The goal of HS00's hybrid scheme, is then to find $\Delta\mathbf{x}$ to minimize (3).

L03 and B05 employ a different approach to incorporate ensemble information in the cost function. They represent the analysis increment as

$$\Delta\mathbf{x} = \beta_1 \Delta\mathbf{x}_1 + \beta_2 \Delta\mathbf{x}_2, \quad (4)$$

$$\Delta\mathbf{x}_1 = (\mathbf{B}_1)^{1/2}\mathbf{v}_1, \quad (5)$$

$$\Delta\mathbf{x}_2 = (\mathbf{B}_2)^{1/2}\mathbf{v}_2, \quad (6)$$

and the associated cost function is

$$J = \frac{1}{2}\mathbf{v}_1^T\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2^T\mathbf{v}_2 + \frac{1}{2}\left(H\left(\mathbf{x}^b + \Delta\mathbf{x}\right) - \mathbf{y}\right)^T \mathbf{R}^{-1}\left(H\left(\mathbf{x}^b + \Delta\mathbf{x}\right) - \mathbf{y}\right). \quad (7)$$

$\mathbf{v}_1$ is a vector of the standard 3D-Var control variables associated with the traditional 3D-Var transform $(\mathbf{B}_1)^{1/2}$. The vector $\mathbf{v}_2$ is the augmented part of the control variable, which is associated with the ensemble covariance. The scalars β_1 and β_2 are the weighting coefficients to combine the two increments $\Delta\mathbf{x}_1$ and $\Delta\mathbf{x}_2$. This choice of control variables also preconditions the background term in (7), as is common in variational methods (e.g., Parrish and Derber 1992; Gauthier et al. 1999; Courtier et al. 1998).

In (6), $(\mathbf{B}_2)^{1/2}$ is the square root of the ensemble covariance. If no covariance localization is applied, $(\mathbf{B}_2)^{1/2}$ is simply the rectangular matrix whose columns are the ensemble perturbations divided by $\sqrt{K-1}$, where K is the ensemble size.

Both L03 and B05 proposed methods to implement covariance localization on the ensemble covariance in a variational system with preconditioning. As shown in the appendix of this note and (B.3) of B05, B05 found the square root of the localized ensemble covariance. Thus the form of the cost function by B05 with covariance localization implemented is still the same as (4-7). After incorporating covariance localization, the cost function of L03 (his eq.17) is a bit different from (6). As shown in the appendix, the cost functions incorporating the correlation matrix by L03 can be manipulated into the same form of B05. Thus in the following proof, for simplicity, we use the general formulation (4-7) to represent the extended control variable method for both L03 and B05. The goal of L03 and B05's hybrid scheme is then to find the control vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to minimize (7) and reconstruct the increment by (4-6).

When $\beta_1 = \sqrt{\alpha_1}$ and $\beta_2 = \sqrt{\alpha_2}$, the hybrid variational methods proposed by HS00 and L03 are mathematically equivalent in the sense that minimizing (3) and (7) produces the same analysis increment. This can be shown as follows:

To find $\Delta\mathbf{x}$ that minimizes (3), we set the first-order derivative of (3) with respect to $\Delta\mathbf{x}$ equal to zero, i.e., $\frac{\partial J}{\partial \Delta\mathbf{x}} = \mathbf{0}$, which gives

$$\Delta\mathbf{x} + \mathbf{B}\mathbf{H}^T\mathbf{R}^{-1}\left(H\left(\mathbf{x}^b + \Delta\mathbf{x}\right) - \mathbf{y}\right) = \mathbf{0}, \tag{8}$$

where $\mathbf{H} \equiv \frac{\partial H(\mathbf{x})}{\partial \mathbf{x}}$, evaluated at the $\mathbf{x}$ that satisfies (8). Solutions for (8) can be found iteratively when the observation operator H is nonlinear. If the observation operator is linear or it is weakly nonlinear and $\mathbf{x}_b$ is reasonably accurate, then explicit solutions can be derived. For details, please refer to Lorenc (1986, 1988), Daley (1991), Parrish and Derber (1992) and Daley and Barker (2001).

Next we find the analysis increment associated with minimizing (7) with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$. To minimize (7), $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy $\frac{\partial J}{\partial \mathbf{v}_1} = \mathbf{0}$ and $\frac{\partial J}{\partial \mathbf{v}_2} = \mathbf{0}$, which gives

$$\mathbf{v}_1 + \left(\mathbf{H}\beta_1\left(\mathbf{B}_1\right)^{1/2}\right)^T \mathbf{R}^{-1}\left(H\left(\mathbf{x}^b + \Delta\mathbf{x}\right) - \mathbf{y}\right) = \mathbf{0}, \tag{9}$$

$$\mathbf{v}_2 + \left(\mathbf{H}\beta_2\left(\mathbf{B}_2\right)^{1/2}\right)^T \mathbf{R}^{-1}\left(H\left(\mathbf{x}^b + \Delta\mathbf{x}\right) - \mathbf{y}\right) = \mathbf{0}, \tag{10}$$

where $\Delta\mathbf{x}$ is given by (4-6). Pre-multiplying (9) by $\beta_1 \mathbf{B}_1^{1/2}$, pre-multiplying (10) by $\beta_2 \mathbf{B}_2^{1/2}$, adding both sides of the subsequent two equations, and using (4), yields

$$\Delta\mathbf{x} + \left(\beta_1^{\,2}\mathbf{B}_1 + \beta_2^{\,2}\mathbf{B}_2\right)\mathbf{H}^T\mathbf{R}^{-1}\left(H\left(\mathbf{x}^b + \Delta\mathbf{x}\right) - \mathbf{y}\right) = \mathbf{0}. \tag{11}$$

So, if $\beta_1 = \sqrt{\alpha_1}$ and $\beta_2 = \sqrt{\alpha_2}$, we can further substitute (2), the HS00 background-error covariance, into (11) and then obtain (8). Consequently, the L03 and B05's increment satisfies the same equation as HS00's increment.

The above proof shows that the analysis increment from (3) and (4-7) will converge to the same solution and thus the two differently proposed hybrid schemes are equivalent.

3. Discussions on operational applications

To apply the HS00 hybrid variational framework (3) operationally, the hybrid background-error covariance given by (2) will need to be preconditioned to speed the minimization process. In comparison, the L03 and B05 hybrid framework (7) may be easier to implement for the variational framework where the preconditioning is with respect to the background term (e.g., Lorenc et al. 2000, Gauthier et al. 1999, Lorenc 2003, Barker et al 2004). In this case, one just needs to extend the traditional control variables and existing preconditioners can be used.

With the approach of L03 and B05, as discussed in the appendix, one can use the ensemble perturbations to construct the square root of the ensemble covariance in (7), assuming an ensemble that is representative of background forecast errors has been generated. The additional control variables then has dimension equal to the ensemble size K. In this case the computational cost incurred by including the extra control variables is small given the ensemble size K is $O(100)$ or less in current operational usage. To reduce the sampling error associated with the limited ensemble size, one can adopt the localizing Schur product proposed by L03 and B05. However, in this case the number of extra control variables is increased to $K \times r$, where r is the rank of the prescribed correlation matrix (see appendix for details and B05). As discussed in B05, and from the experiments implementing the L03 framework within UK Met Office variational assimilation (Barker 1999) and the Weather Research and Forecast Model variational assimilation (Dale Barker, personal communication 2005), using a spectrally truncated expansion to represent the prescribed correlation matrix can reduce the cost.

The method proposed in L03 and B05 to incorporate the ensemble covariance by augmenting the control variables is particularly suitable for model-space variational schemes that precondition with respect to the background term (e.g., Lorenc et al. 2000, Gauthier et al. 1999, Lorenc 2003, Barker et al. 2004). For observation-space schemes, such as the Naval Research Laboratory Atmospheric Variational Data Assimilation System (NAVDAS; Daley and Barker 2001), an observation-space preconditioner is used and thus no square root of the model-space background covariance matrix is required. Such systems can hybridize the ensemble covariance by directly linearly combining the ensemble covariance with the standard 3DVAR covariance (Craig H. Bishop, personal communication, 2005).

The next question is how to find an appropriate ensemble. Perhaps short-term forecasts from operational systems could be used as the background ensemble in (7). However, the improvement in hybrid analysis accuracy over a more standard variational approach may depend substantially upon the specific method of ensemble generation. Ideally, the ensemble should be drawn from the distribution of background forecast errors and will depend on the prior observations and the chaotic error growth of the day. Operational ensemble techniques may not be optimized for this application (e.g., Hamill et al. 2000). Another candidate for the ensemble generation may be the ensemble transform Kalman filter (ETKF; Bishop et al. 2001; Wang and Bishop 2003; Wang et al. 2004), which has been shown to provide inexpensive but relatively skillful ensemble forecasts and is designed specifically to produce forecasts that realistically account for the error reduction by assimilation of observations and the subsequent growth of errors during the forecast. Recent work comparing the hybrid ETKF-3DVAR and the ensemble square root filter (EnSRF) suggests that the hybrid ETKF-3DVAR can achieve a large portion of the improvement of the EnSRF over the 3DVAR (Wang et al. 2005).

4. Conclusions

In hybrid ensemble/variational data assimilation schemes, ensemble covariances that reflect flow-dependent forecast-error uncertainty are incorporated into the variational framework. Methods have been proposed to achieve this. In Hamill and Snyder (2000), the background error covariance was defined explicitly as a linear combination of the standard 3DVAR covariance and the ensemble covariance. In Lorenc (2003) and Buehner (2005), the original variational control variables were extended by another set of control variables preconditioned upon the square root of the ensemble covariance. They also suggested how to incorporate a localizing Schur product to the variational framework with preconditioning. Here we have demonstrated that the hybrid schemes proposed by Hamill and Snyder (2000), Lorenc (2003) and Buehner (2005) are mathematically equivalent. Assuming a suitable ensemble has been constructed, the Lorenc (2003) and Buehner (2004) framework should be easier to apply in model-space variational schemes where preconditioning is performed with respect to the background term. For operational centers that run ensemble forecasts and variational data assimilation, the hybrid scheme may provide an effective and feasible way to improve the analysis without the cost of a full implementation of an ensemble-based data assimilation approach.

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Appendix

On the equivalence of Lorenc (2003) and Buehner (2005) in implementing *localized* ensemble covariance in the variational framework with preconditioning

Denote $\mathbf{X}^f = \left(\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_K\right)$ as the deviation from the ensemble mean normalized by $\sqrt{K-1}$, where K is the ensemble size. The sample ensemble covariance is $\mathbf{P} = \mathbf{X}^f\left(\mathbf{X}^f\right)^T$. Thus, if no covariance localization is applied, in (6), $\left(\mathbf{B}_2\right)^{1/2} = \mathbf{X}^f$, an $N \times K$ rectangular matrix, where N is the dimension of the state vector, and $\mathbf{v}_2$ is a vector of K elements.

Further denote $\mathbf{S}$ as the prescribed correlation matrix used for covariance localization. Then the localized ensemble covariance is the Schur product of $\mathbf{P}$ and $\mathbf{S}$, i.e., $\mathbf{P} \circ \mathbf{S}$. In order to match this localized ensemble covariance in the variational framework with preconditioning, B05 modified (6) as followed. First $\left(\mathbf{B}_2\right)^{1/2}$ is defined as

$$\left(\mathbf{B}_2\right)^{1/2} = \left[diag\left(\mathbf{x}'_1\right)\mathbf{S}^{1/2}, diag\left(\mathbf{x}'_1\right)\mathbf{S}^{1/2}, \ldots, diag\left(\mathbf{x}'_K\right)\mathbf{S}^{1/2}\right], \tag{A.1}$$

where $diag\left(\mathbf{x}'_k\right)$, $k = 1, \ldots, K$, represents a matrix with vector $\mathbf{x}'_k$ along its diagonal. It was shown in B05 that (A.1) satisfies $\left(\mathbf{B}_2\right)^{1/2}\left[\left(\mathbf{B}_2\right)^{1/2}\right]^T = \mathbf{P} \circ \mathbf{S}$. The associated extended control variables are defined as

$$\mathbf{v}_2 = \begin{pmatrix} \mathbf{v}_{21} \\ \mathbf{v}_{22} \\ \vdots \\ \mathbf{v}_{2K} \end{pmatrix}, \tag{A.2}$$

where $\mathbf{v}_{2k}$, $k = 1,\ldots,K$, is a vector of r elements. Note r is the rank of $\mathbf{S}$. With other terms in (7) unchanged, the second term of the cost function is then given by the inner product of (A.2) and the $\Delta\mathbf{x}_2$ term in (6) is given by (A.1) times (A.2) instead. B05's cost function with covariance localization has the same form as (4-7).

L03 incorporated $\mathbf{S}$ in the cost function in a different form (see eq. 17 of L03). The second term of the cost function is redefined as

$$J_2 = \frac{1}{2}\mathbf{a}^T \begin{pmatrix} \mathbf{S} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{S} \end{pmatrix}^{-1} \mathbf{a}, \tag{A.3}$$

where the block diagonal matrix is constructed by listing K correlation matrices $\mathbf{S}$. In (A.3) the newly defined extended control variables are

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_K \end{pmatrix}, \tag{A.4}$$

and each of $\mathbf{a}_k$, $k = 1,\cdots,K$, is a vector of N elements. The $\Delta\mathbf{x}_2$ term is modified as

$$\Delta\mathbf{x}_2 = \left(\mathbf{X}^f \circ \mathbf{A}\right)\mathbf{1}, \tag{A.5}$$

where $\mathbf{A} = \left(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_K\right)$, $\mathbf{X}^f \circ \mathbf{A}$ is the Schur product of $\mathbf{X}^f$ and $\mathbf{A}$, and $\mathbf{1}$ is a vector of K elements that are all equal to one.

Next we show that by linearly transforming $\mathbf{a}$, L03's cost function with covariance localization incorporated, can be manipulated into the same format as that of B05. Thus they will lead to the same solution.

We further define a new set of extended control variables $\mathbf{v}_2$, which are linearly related to $\mathbf{a}$ by

$$\mathbf{a} = \begin{pmatrix} \mathbf{S}^{1/2} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{S}^{1/2} \end{pmatrix} \mathbf{v}_2, \tag{A.6}$$

where $\mathbf{v}_2$ is given by (A.2). Substituting (A.6) into (A.3), then the second term of the cost function becomes the inner product of $\mathbf{v}_2$, the same as B05.

Further substituting (A.6) into $\mathbf{A}$, we obtain

$$\mathbf{A} = \mathbf{S}^{1/2}\mathbf{V}, \tag{A.7}$$

where $\mathbf{V} = \left(\mathbf{v}_{21}, \mathbf{v}_{22}, \ldots, \mathbf{v}_{2K}\right)$. Thus (A.5) becomes

$$\Delta\mathbf{x}_2 = \left[\mathbf{X}^f \circ \left(\mathbf{S}^{1/2}\mathbf{V}\right)\right]\mathbf{1}. \tag{A.8}$$

Next we need to show that, $\Delta\mathbf{x}_2$, defined as (A.1) times (A.2) by B05 and as (A.8) by L03 are the same.

Denoting $\mathbf{V} = \left(v_{ij}\right)$, $i = 1, \ldots, r$, $j = 1, \ldots, K$; $\mathbf{S}^{1/2} = \left(s_{ij}\right)$, $i = 1, \ldots, N$, $j = 1, \ldots, r$; $\mathbf{X}^f = \left(x_{ij}\right)$, $i = 1, \ldots, N$, $j = 1, \ldots, K$; $\mathbf{A} = \mathbf{S}^{1/2}\mathbf{V} = \left(a_{ij}\right)$, $i = 1, \ldots, N$, $j = 1, \ldots, K$., and writing (A.8) in element format, we obtain the *i*th element of $\Delta\mathbf{x}_2$ by L03 as

$$\left(\Delta\mathbf{x}_2\right)_i = \sum_{j=1}^{K} x_{ij} a_{ij} = \sum_{j=1}^{K} x_{ij} \sum_{m=1}^{r} s_{im} v_{mj} = \sum_{j=1}^{K} \sum_{m=1}^{r} x_{ij} s_{im} v_{mj}. \tag{A.9}$$

Substituting (A.1) and (A.2) into (6), we obtain $\Delta\mathbf{x}_2$ by B05 as

$$\Delta\mathbf{x}_2 = \sum_{k=1}^{K} \left(diag\left(\mathbf{x}_k\right)\mathbf{S}^{1/2}\right)\mathbf{v}_{2k}. \tag{A.10}$$

Denote $\mathbf{D}_k = diag\left(\mathbf{x}_k\right)\mathbf{S}^{1/2} = \left(d_{ij}\right)_k$, $i = 1, \ldots, N$, $j = 1, \ldots, r$, $k = 1, \ldots, K$, and note that $\left(d_{ij}\right)_k = s_{ij} x_{ik}$. Substituting $\left(d_{ij}\right)_k$ into (A.10) and writing in element format, we obtain the *i*th element of $\Delta\mathbf{x}_2$ by B05 as

$$\left(\Delta \mathbf{x}_2\right)_i = \sum_{k=1}^{K}\left(\sum_{j=1}^{r}\left(d_{ij}\right)_k v_{jk}\right) = \sum_{k=1}^{K}\left(\sum_{j=1}^{r} s_{ij} x_{ik} v_{jk}\right) = \sum_{k=1}^{K}\sum_{j=1}^{r} s_{ij} x_{ik} v_{jk}\,. \tag{A.11}$$

From (A.9) and (A.11), the $\Delta \mathbf{x}_2$ terms in L03 and B05, with the localized ensemble covariance incorporated, are the same.

To summarize, the above shows that after linear transformation on the extended control variables, L03's cost function with covariance localization applied has the same form as B05. In other words, it can be written as (4-7).

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